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On the calculation of group characters

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Abstract

It is known that characters of irreducible representations of finite Lie algebras can be obtained using the Weyl character formula including Weyl group summations which make actual calculations almost impossible except for a few Lie algebras of lower rank. By starting from the Weyl character formula, we show that these characters can be re-expressed without referring to Weyl group summations. Some useful technical points are given in detail for the instructive example of G_2 Lie algebra.

Keywords: Characters of Lie algebras; G_2 Lie algebra

1. Introduction

It is known that the character formula of Weyl [1] gives us a direct way to calculate the character of irreducible representations of finite Lie algebras. For this, let G_r be a Lie algebra of rank r, let $W(G_r)$ be its Weyl group, and let the α_i's and λ_i's be, respectively, its simple roots and fundamental dominant weights. The notation here and in the following sections will be as in our previous work [2]. For further reading, we refer the reader to the beautiful book of Humphreys [3]. A dominant weight Λ^+ is expressed in the form

$$\Lambda^+ = \sum_{i=1}^{r} s_i \lambda_i \tag{1.1}$$

where the s_i's are some positive integers (including zero). An irreducible representation $V(\Lambda^+)$ can then be attributed to Λ^+. The character $Ch(\Lambda^+)$ of $V(\Lambda^+)$ is defined by

$$Ch(\Lambda^+) \equiv \sum_{\lambda^+} \sum_{\mu \in W(\lambda^+)} m_{\Lambda^+}(\mu) \mathrm{e}^{\mu} \tag{1.2}$$

where the $m_{\Lambda^+}(\mu)$'s are multiplicities which count the number of times that a weight μ is repeated for $V(\Lambda^+)$. The first sum here is over Λ^+ and all of its sub-dominant weights λ^+'s while the second sum is over the elements of their Weyl orbits $W(\lambda^+)$'s. Formal exponentials are taken just as in the book of Kac [4] and in (1.2) we extend the concept to any weight μ, in the form e^{μ}. Note here that multiplicities are invariant under Weyl group actions and hence it is sufficient to determine only $m_{\Lambda^+}(\lambda^+)$ for the whole Weyl orbit $W(\lambda^+)$.

An equivalent form of (1.2) can be given as

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$$Ch(\Lambda^+) = \frac{A(\rho + \Lambda^+)}{A(\rho)} \tag{1.3}$$

where ρ is the Weyl vector of G_r. (1.3) is the celebrated *Weyl character formula* which gives us the possibility of calculating characters in the most direct and efficient way. The central objects here are $A(\rho + \Lambda^+)$'s which include a sum over the whole Weyl group:

$$A(\rho + \Lambda^+) \equiv \sum_{\sigma \in W(G_r)} \epsilon(\sigma) \mathrm{e}^{\sigma(\rho+\Lambda^+)}. \tag{1.4}$$

In (1.4), σ denotes an element of the Weyl group, i.e. a Weyl reflection, and $\epsilon(\sigma)$ is the corresponding *signature* with values either $+1$ or -1.

The structure of Weyl groups is completely known for finite Lie algebras in principle. In practice, however, the problem is not so trivial, especially for Lie algebras of some higher rank. The order of the E_8 Weyl group is, for instance, 696 729 600 and any application of the Weyl character formula needs for E_8 an explicit calculation of a sum over 696 729 600 Weyl reflections. Our main point here is overcoming this difficulty in an essential manner.

For an actual application of (1.3), an important thing to notice is the *specialization* of formal exponentials e^{μ} as it is called in the book of Kac [4]. In its most general form, we consider here the specialization

$$\mathrm{e}^{\alpha_i} \equiv u_i, \quad i = 1, 2, \ldots, r. \tag{1.5}$$

which allows us to obtain $A(\rho)$ in the form of

$$A(\rho) = P(u_1, u_2, \ldots, u_r) \tag{1.6}$$

where $P(u_1, u_2, \ldots, u_r)$ is a polynomial in indeterminates u_i. We also have

$$A(\rho + \Lambda^+) = P(u_1, u_2, \ldots, u_r; s_1, s_2, \ldots, s_r) \tag{1.7}$$

where $P(u_1, u_2, \ldots, u_r; s_1, s_2, \ldots, s_r)$ is another polynomial of indeterminates u_i and also parameters s_i defined in (1.1). The Weyl formula (1.3) then says that polynomial (1.7) always factorizes in polynomial (1.6) and also another polynomial $R(u_1, u_2, \ldots, u_r; s_1, s_2, \ldots, s_r)$ which is nothing but the character polynomial of $V(\Lambda^+)$.

The specialization (1.5) will always be normalized in such a way that (1.3) gives us the *Weyl dimension formula* [5], in the limit $u_i = 1$ for all $i = 1, 2, \ldots, r$. One also expects that

$$P(u_1, u_2, \ldots, u_r; 0, 0, \ldots, 0) \equiv P(u_1, u_2, \ldots, u_r). \tag{1.8}$$

2. Recreating $A(\rho + \Lambda^+)$ from $A(\rho)$

In this section, without any reference to Weyl groups, we give a way to calculate polynomial (1.7) directly from polynomial (1.6). For this, we first give the following explicit expression for polynomial (1.6):

$$A(\rho) = \frac{\prod_{\alpha \in \Phi^+} (\mathrm{e}^{\alpha} - 1)}{\prod_{i=1}^{r} (\mathrm{e}^{\alpha_i})^{k_i}} \tag{2.1}$$

where

$$k_i \equiv \frac{1}{2}(\alpha_i, \alpha_i)(\lambda_i, \rho) \tag{2.2}$$

and Φ^+ is the positive root system of G_r. Exponents k_i are due to the fact that the monomial of maximal order is

$$\prod_{i=1}^{r} (\mathrm{e}^{\alpha_i})^{2k_i}$$

in the product $\prod_{\alpha \in \Phi^+} (\mathrm{e}^{\alpha} - 1)$. All the scalar products like (λ_i, α_j) or (α_i, α_i) are symmetrical ones and they are known to be defined via the Cartan matrix of a Lie algebra. The crucial point, however, is seeing that (2.1) is equivalent to

$$A(\rho) = \prod_{A=1}^{|W(G_r)|} \epsilon_A (\mathrm{e}^{\alpha_i})^{\xi_i^0(A)} \tag{2.3}$$

where $|W(G_r)|$ is the order of the Weyl group $W(G_r)$ and, as is emphasized in Section 1, the ϵ_A's are signatures with values $\epsilon_A = \mp 1$. Note here that, by expanding the product $\prod_{\alpha\in\Phi^+}(\mathrm{e}^{\alpha}-1)$, the precise values of signatures can be determined uniquely.

To get an explicit expression for exponents $\xi_i^0(A)$ in (2.3), let us define R^+ as composed from elements of the form

$$\beta^+ \equiv \sum_{i=1}^{r} n_i \alpha_i \tag{2.4}$$

where the n_i's are some positive integers including zero. R^+ is a subset of the positive root lattice of G_r. The main emphasis here is on some special roots

$$\gamma_i(I_i) \in R^+$$

which are defined by following conditions:

$$(\lambda_i - \gamma_i(I_i), \lambda_j - \gamma_j(I_j)) = (\lambda_i, \lambda_j), \quad i, j = 1, 2, \ldots, r. \tag{2.5}$$

Note here that the (λ_i, λ_j)'s are defined by the inverse Cartan matrix. For the range of indices I_j, we assume that they take values from the set $\{1, 2, \ldots, |I_j|\}$, that is

$$I_j \in \{1, 2, \ldots, |I_j|\}, \quad j = 1, 2, \ldots, r.$$

We also define the sets

$$\Gamma(A) \equiv \{\gamma_1(I_1(A)), \gamma_2(I_2(A)), \ldots, \gamma_r(I_r(A))\}, \quad A = 1, 2, \ldots, D \tag{2.6}$$

chosen using conditions (2.5) where $I_j(A) \in \{1, 2, \ldots, |I_j|\}$ and D is the maximal number of these sets.

The following two statements are then valid:

(1) $\mathbf{D} = |\mathbf{W}(\mathbf{G_r})|$
(2) $|\mathbf{I_j}| = |\mathbf{W}(\boldsymbol{\lambda}_\mathbf{j})|$

where $|W(\lambda_j)|$ is the order, i.e. the number of elements, of the Weyl orbit $W(\lambda_j)$. As is known, a Weyl orbit is stable under Weyl reflections and hence all its elements have the same length. It is interesting to note however that the lengths of any two elements $\gamma_i(j_1)$ and $\gamma_i(j_2)$ could, in general, be different while the statement (2) is, still, valid.

The exponents in (2.3) can now be defined by

$$\xi_i^0(A) \equiv \frac{1}{2}(\alpha_i, \alpha_i)(\lambda_i - \gamma_i(A), \rho). \tag{2.7}$$

We then also state that the natural extension of (2.3) is as follows:

$$A(\rho + \Lambda^+) = \prod_{A=1}^{|W(G_r)|} \epsilon_A (\mathrm{e}^{\alpha_i})^{\xi_i(A)} \tag{2.8}$$

for which the exponents are

$$\xi_i(A) \equiv \frac{1}{2}(\alpha_i, \alpha_i)(\lambda_i - \gamma_i(A), \rho + \Lambda^+) \tag{2.9}$$

as a natural extension of (2.7). Signatures have the same value in both expressions, (2.3) and (2.8). This reduces the problem of explicit calculation of the character polynomial $Ch(\Lambda^+)$ to the problem of finding solutions to conditions (2.5). It is clear that this is more manageable than using the Weyl character formula directly.

3. An example: G_2

It is known that G_2 is characterized by two different root lengths

$$(\alpha_1, \alpha_1) = 6, \qquad (\alpha_2, \alpha_2) = 2 \tag{3.1}$$

and also

$$(\alpha_1, \alpha_2) = -3. \tag{3.2}$$

Its positive root system is thus

$$\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}. \tag{3.3}$$

Beyond Φ^+, the first few elements of R^+ are determined by

$$\begin{aligned}
&2\alpha_2, 2\alpha_1 + 2\alpha_2, 2\alpha_1 + 4\alpha_2, \alpha_1 + 4\alpha_2,\\
&2\alpha_1 + \alpha_2, 2\alpha_1 + 5\alpha_2, 3\alpha_1 + 4\alpha_2, 3\alpha_1 + 5\alpha_2, 3\alpha_2,\\
&3\alpha_1 + 3\alpha_2, 3\alpha_1 + 6\alpha_2, 2\alpha_1, 2\alpha_1 + 6\alpha_2, 4\alpha_1 + 6\alpha_2
\end{aligned}$$

as will be seen from the definition (2.4). This will be sufficient for obtaining the following elements which fulfill conditions (2.5):

$$\begin{aligned}
\gamma_1(1) &= 0,\\
\gamma_1(2) &= \alpha_1,\\
\gamma_1(3) &= \alpha_1 + 3\alpha_2,\\
\gamma_1(4) &= 3\alpha_1 + 3\alpha_2,\\
\gamma_1(5) &= 3\alpha_1 + 6\alpha_2,\\
\gamma_1(6) &= 4\alpha_1 + 6\alpha_2
\end{aligned}$$

and

$$\begin{aligned}
\gamma_2(1) &= 0,\\
\gamma_2(2) &= \alpha_2,\\
\gamma_2(3) &= \alpha_1 + \alpha_2,\\
\gamma_2(4) &= \alpha_1 + 3\alpha_2,\\
\gamma_2(5) &= 2\alpha_1 + 3\alpha_2,\\
\gamma_2(6) &= 2\alpha_1 + 4\alpha_2
\end{aligned}$$

from which we obtain, as in (2.6), the following 12 sets of 2-elements:

$$\begin{aligned}
\Gamma_1 &= \{\gamma_1(1), \gamma_2(1)\}, & \epsilon_1 &= +1\\
\Gamma_2 &= \{\gamma_1(2), \gamma_2(3)\}, & \epsilon_2 &= +1\\
\Gamma_3 &= \{\gamma_1(3), \gamma_2(2)\}, & \epsilon_3 &= +1\\
\Gamma_4 &= \{\gamma_1(4), \gamma_2(5)\}, & \epsilon_4 &= +1\\
\Gamma_5 &= \{\gamma_1(5), \gamma_2(4)\}, & \epsilon_5 &= +1\\
\Gamma_6 &= \{\gamma_1(6), \gamma_2(6)\}, & \epsilon_6 &= +1\\
\Gamma_7 &= \{\gamma_1(1), \gamma_2(2)\}, & \epsilon_7 &= -1\\
\Gamma_8 &= \{\gamma_1(2), \gamma_2(1)\}, & \epsilon_8 &= -1\\
\Gamma_9 &= \{\gamma_1(3), \gamma_2(4)\}, & \epsilon_9 &= -1\\
\Gamma_{10} &= \{\gamma_1(4), \gamma_2(3)\}, & \epsilon_{10} &= -1\\
\Gamma_{11} &= \{\gamma_1(5), \gamma_2(6)\}, & \epsilon_{11} &= -1\\
\Gamma_{12} &= \{\gamma_1(6), \gamma_2(5)\}, & \epsilon_{12} &= -1.
\end{aligned} \tag{3.4}$$

Note here that $|W(G_2)| = 12$, $|W(\lambda_1)| = 6$ and $|W(\lambda_2)| = 6$ show, for G_2, the validity of our two statements mentioned above. The signatures are given on the right-hand side of (3.4). They can be easily found to be true by comparing the two expressions (2.1) and (2.3) for the case in hand. As in (1.5), let us choose the most general specialization

$$e^{\alpha_1} = x, \qquad e^{\alpha_2} = y.$$

Then, in view of (1.8), the character polynomial

$$R(x, y, s_1, s_2) \equiv \frac{P(x, y, s_1, s_2)}{P(x, y, 0, 0)} \tag{3.5}$$

for an irreducible G_2 representation originated from a dominant weight $\Lambda^+ = s_1\lambda_1 + s_2\lambda_2$ will be expressed in the following *ultimate* form:

$$\begin{aligned} P(x, y, s_1, s_2) = &+x^{3+2s_1+s_2}y^{5+3s_1+2s_2} + x^{-3-2s_1-s_2}y^{-5-3s_1-2s_2} + x^{-2-s_1-s_2}y^{-1-s_2} + x^{2+s_1+s_2}y^{1+s_2} \\ &+ x^{-1-s_1}y^{-4-3s_1-s_2} + x^{1+s_1}y^{4+3s_1+s_2} - x^{1+s_1}y^{-1-s_2} - x^{-1-s_1}y^{1+s_2} \\ &- x^{-2-s_1-s_2}y^{-5-3s_1-2s_2} - x^{2+s_1+s_2}y^{5+3s_1+2s_2} - x^{-3-2s_1-s_2}y^{-4-3s_1-s_2} \\ &- x^{3+2s_1+s_2}y^{4+3s_1+s_2}. \end{aligned} \tag{3.6}$$

In the following we will give some examples to illustrate (3.5);

$$\begin{aligned} Ch(\lambda_1) \;=\; & R(x, y, 1, 0) = \frac{1}{x^2y^3}(1 + x + xy + xy^2 + x^2y^2 + xy^3 + 2x^2y^3 + x^3y^3 \\ & + x^2y^4 + x^3y^4 + x^3y^5 + x^3y^6 + x^4y^6) \end{aligned}$$

$$Ch(\lambda_2) = R(x, y, 0, 1) = \frac{1}{xy^2}(1 + y + xy + xy^2 + xy^3 + x^2y^3 + x^2y^4)$$

$$Ch(\lambda_1 + \lambda_2) = R(x, y, 1, 1) = \frac{1}{x^3y^5}(1 + x)(1 + y)(1 + xy)(1 + xy^2)(1 + xy^3)(1 + x^2y^3)$$

$$Ch(2\lambda_2) = R(x, y, 0, 2) = \frac{1}{x^2y^4}(1 + y + y^2)(1 + xy + x^2y^2)(1 + xy^2 + x^2y^4)$$

where

$$P(x, y, 0, 0) = \frac{1}{x^3y^5}(1 - x)(1 - y)(1 - xy)(1 - xy^2)(1 - xy^3)(1 - x^2y^3).$$

A useful application of (3.5) is in the calculation of tensor coupling coefficients [6]. We note, for instance, that the coefficients in the decomposition

$$\begin{aligned} V(\lambda_1) \otimes V(\lambda_1 + \lambda_2) \;=\; & V(2\lambda_1 + \lambda_2) \oplus V(\lambda_1 + 2\lambda_2) \oplus V(4\lambda_2) \oplus V(3\lambda_2) \\ & \oplus 2V(\lambda_1 + \lambda_2) \oplus V(2\lambda_2) \oplus V(\lambda_2) \end{aligned}$$

can be calculated easily using

$$\begin{aligned} R(x, y, 1, 0) \otimes R(x, y, 1, 1) \;=\; & R(x, y, 2, 1) \oplus R(x, y, 1, 2) \oplus R(x, y, 0, 4) \oplus R(x, y, 0, 3) \\ & \oplus 2R(x, y, 1, 1) \oplus R(x, y, 0, 2) \oplus R(x, y, 0, 1) \end{aligned}$$

where the terms above are already known by (3.5).

4. Conclusion

Although the application of our method is given here for the case of G_2 Lie algebra, there are no serious difficulties in doing the same for classical chains A_r, B_r, C_r and D_r of finite Lie algebras. Our results for exceptional Lie algebras F_4 and E_6 will be given soon in a subsequent publication. In the case of E_7 and E_8 Lie algebras, one must however emphasize that rather than the most general specialization (1.5) of formal exponentials we need in practice simpler specializations though there are some other methods [7] which work equally well for E_7 and E_8. For affine Lie algebras, a similar study of the Weyl–Kac character formula will also be interesting.

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